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Strong Unicity of Arbitrary Rate

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The concept is introduced of strong unicity with respect to a rate function u, i.e., $\|f-p\| \geqslant \|f-p_f\| + \gamma u(\|p-p_f\|)$, in approximating (with constraints) f in a Banach space X from an n-dimensional subspace V ($p \in V$, p_f denotes the best approximation to f, and γ denotes a positive constant). Past work has demonstrated examples of monotone approximation in $C[a,b]$, where V is Haar and the best u has polynomial decay of arbitrary even degree (i.e., $u(t)=t^{2m}$, $m=1,2,\ldots,$). In particular, in this same setting examples are demonstrated where the best u decays exponentially (e.g., $\exp(-c_2 t^{-2/3}) \leqslant u(t) \leqslant t^{-2/3}\exp(-c_1 t^{-2/3})$ for constants $0<c_1<c_2$) and a general statement is provided relating the best u to h'' when $V=[1, x, h'(x), h(x)]$ and $h \in C^2$ satisfies certain conditions.

From [4] and [5] we have the existence of $f \in C[a,b]$ such that, if p_f denotes a best (monotone) approximation to f from $M_4 = V_4 \cap \{p : p'(x) \geqslant 0\}$, where $V_4 = [1, x, x^2, x^3]$, then p_f is *strongly unique of order* $\frac{1}{2}$ ([5]) and the order $\frac{1}{2}$ is best possible ([4]). That is, for each $N>0$ there is a constant $\gamma > 0$ such that, for all $p \in M_4$ with $\|p\| \leqslant N$,

$$\|f-p\| \geqslant \|f-p_f\| + \gamma \|p-p_f\|^{1/\alpha}, \tag{1}$$

where $\alpha = \frac{1}{2}$ and is best possible (i.e., no larger α will suffice).

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In [3] the above result is extended to the cases where $V_4 = [1, x, x^{2m}, x^{2m+1}]$, where $m = 1, 2, \ldots$; i.e., (1) holds where the "order" $\alpha = 1/2m$ and is best possible.

DEFINITION 1. If p_f is a best uniform approximation to $f \in C[a, b]$ from W, a subset of $C[a, b]$, we shall say that p_f is *strongly unique with* (respect to the) *rate* (function) u ($u \in C[0, \infty)$, u is increasing, and $u(0) = 0$) if for each $N > 0$ there is a constant $\gamma > 0$ such that

$$\|f - p\| \geqslant \|f - p_f\| + \gamma u(\|p - p_f\|) \tag{2}$$

for all $p \in W$ satisfying $\|p\| \leqslant N$. We shall say that the *rate* (of strong unicity) *is at best* u if (2) cannot be satisfied by any u_1, where $u(x) = o(u_1(x))$, $x \to 0^+$.

EXAMPLE 1. For the cases treated in [3], $u(x) = \kappa x^{2m}$ (for an arbitrary constant $\kappa > 0$). Note in fact that $u(x) = x(x^{2m+1})''$, where $x^{2m+1} \in V_4 = [1, x, x^{2m}, x^{2m+1}]$. We thus have an example of the following two theorems by taking $h(x) = x^{2m+1}$ and $\varphi(x) = x$ and noting that $(h'/h'')/\varphi = 1/2m < 1/(2m-1) = (h''/h''')/\varphi$.

THEOREM 1. *Take $V_4 = [1, x, h'(x), h(x)]$ to be a Haar space in some neighborhood $(-\alpha, \alpha)$ of the origin, where $h \in C^2(-\infty, \infty)$, h is odd, $h'(0) = 0$, h'' is strictly increasing, and $h'(x)/h''(x)$ is asymptotic (as $x \to 0^+$) to $\lambda\varphi(x)$, $\lambda > 0$, where $\varphi \in C[0, \infty)$, $\varphi(0) = 0$, and φ strictly increases to ∞. Then if we take $W = M_4$ (i.e., monotone approximation from V_4), there is an $f \in C[a, b]$, $0 \in (a, b) \subset (-\alpha, \alpha)$ such that the best approximation p_f to f is unique and the rate of strong unicity is at best $u(x) = xh''(\varphi^{-1}(cx))$ for some constant $c > 0$. Furthermore, f can itself be chosen monotone.*

THEOREM 2. *In addition to the hypotheses of Theorem 1, suppose $h \in C^3(0, \infty)$, $\varphi \in C^1[0, \infty)$, $\varphi'(x) > 0$ for $x > 0$, and $\lambda\varphi'(0) < 1$. Then, for f in Theorem 1, p_f is strongly unique with respect to $u(x) = x\psi(y)\, h''(y)$, where $y = \varphi^{-1}(cx)$ for some constant $c > 0$, and*

(i) *if $\varphi'(0) > 0$, $\psi \equiv 1$;*

(ii) *if $\varphi'(0) = 0$, ψ is any positive nondecreasing continuous function asymptotic to $[(h''/h''') - (h'/h'')]/\varphi$.*

Note. If $\varphi'(0) > 0$, then $[(h''/h''') - (h'/h'')]/\varphi$ is asymptotic to a positive constant; thus (i) and (ii) can be combined and replaced by "ψ is any positive continuous function asymptotic to $[(h''/h''') - (h'/h'')]/\varphi$."

Remark. That h is odd and continuous implies $h(0)=0$; $h \in C^1$ implies h' is even; $h \in C^2$ implies h'' is odd and $h''(0)=0$; h'' strictly increasing and $h'(0)=0$ implies $h'(x) \geqslant 0$ with equality only for $x=0$. Likewise, when $h \in C^3(0, \alpha)$, we have on $(-\alpha, 0) \cup (0, \alpha)$ that h''' is even and $\geqslant 0$ with equality not occurring on an interval.

For the proofs of the theorems, we need a lemma. Let l_3 denote the linear functional on V_4 assigning to $p(x)=\alpha_1+\alpha_2 x+\alpha_3 h'(x)+\alpha_4 h(x)$ the number α_3.

LEMMA. *Any set* $\{e_{x_1}, e_{x_2}, e'_0, l_3\}$, $x_1 \neq x_2$, *is independent in* V_4^*. (*Notation*: $e_x(f)=f(x)$ *and* $e'_x(f)=f'(x)$.)

Proof. Suppose $l=a_1 e_{x_1}+a_2 e_{x_2}+a_3 e'_0+a_4 l_3=0 \in V_4^*$, and consider the 4×4 matrix equation obtained by evaluating l at 1, x, h, h', respectively:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ h(x_1) & h(x_2) & 0 & 0 \\ h'(x_1) & h'(x_2) & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

But the determinant of the matrix is easily seen to be $[h(x_1)-h(x_2)] \neq 0$ since h is increasing. Thus $a_i=0$, $1 \leqslant i \leqslant 4$. ∎

Proof of Theorems 1 *and* 2. The proof is a generalization of the techniques of [3]. Since V_4 is Haar, for any $r_1<r_2<r_3$ in $(-\alpha, \alpha)$ with $0 \in (r_2, r_3)$ there is a unique (up to a nonzero scalar multiple) nonzero $p_0 \in V_4$ vanishing at r_1, r_2, r_3. Hence there is a point $\xi=\xi(r_2, r_3) \in (r_2, r_3)$ such that $p'_0(\xi)=0$. Now if $\xi<0$, then move r_2 towards 0 continuously from the left; clearly, by the continuity, for some r_2, $\xi=0$. Similarly, if $\xi>0$, then move r_3 towards 0 continuously from the right; clearly, for some r_3, $\xi=0$. We conclude that there exists a nonzero $p_0 \in V_4$ vanishing at some $r_1<r_2<r_3$ in $(-\alpha, \alpha)$ and such that $p'_0(0)=0$, where $0 \in (r_2, r_3)$. We can therefore take $p_0=h+\kappa_1 h'+c_0$, where $c_0 \neq 0$. Set $[a, b]=[r_1, r_3]$. Now define g to be the 4-piece piecewise linear function joining the five points $(r_i, (-1)^i)$, $i=1,2,3$, and $(\pm\varepsilon, 0)$, where ε is fixed so that $r_2<-\varepsilon<0<\varepsilon<r_3$, and define $f=g+\kappa h$, where κ is a positive constant to be determined later. We now show that κh is a best approximation to f (see, e.g., [2]) by noting that $\{-e_{r_1}, e_{r_2}, -e_{r_3}, e'_0\}$ is an extremal set for f and κh whose convex hull contains the zero of V_4^*, as follows: From the existence of p_0 we have that $l=\lambda_1(-e_{r_1})+\lambda_2 e_{r_2}+\lambda_3(-e_{r_3})+\lambda_4 e'_0=0 \in V_4^*$ for some choice of $\{\lambda_i\}_{i=1}^4$.

But by the lemma, $V_4^0 = V_4 \cap \{p : p'(0) = l_3(p) = 0\}$ is a two-dimensional Haar space and thus by restricting l to V_4^0, we see that all λ_i $(i = 1, 2, 3)$ are of the same nonzero sign since, as is well known, "ordinary alternation" occurs in Haar spaces. We need only show, therefore, that $\lambda_3 \lambda_4 > 0$. But now let $p(x) \in V_4$ have zeros at r_1, r_2, and 0 and satisfy $p(r_3) = 1$. Clearly, $p'(0) \geqslant 0$. If $p'(0) = 0$, then $l(p) = 0$ would imply that $\lambda_3 = 0$, which is not possible. Thus $p'(0) > 0$ and $p'(0) p(r_3) > 0$. Hence $\lambda_4 \lambda_3 > 0$. Thus κh is a best approximation to f and, by referring to the general theory of [1], we can easily see that κh is a unique best approximation, as follows: First, by L'Hospital's rule, note that $\lim_{x \to 0} h(x)/h'(x) = \lim_{x \to 0} h'(x)/h''(x) = 0$ and so h' dominates h near 0. Next note that whenever e_0' is an extremal functional for a best approximation p_f, then $p_f = a_1 + a_3 h' + a_4 h$ and hence also $l_3(p_f) = 0$ (otherwise, nonmonotonic h' dominates h near 0), so that l_3 is an *augmented extremal*. Thus, by the lemma, V_4 is *generalized Haar with respect to f and κh* (see [1] for definition) and we conclude by the theory of [1] that κh is the unique best approximation to f. We therefore can write unambiguously p_f for κh.

We now show that the rate (of strong unicity) is u at best. Define $p_\alpha(x) = p_f(x) + \alpha[p_0(x) + \kappa_1 h''(\varphi^{-1}(\alpha)) x]$ for $0 < \alpha \leqslant \alpha_0$, where α_0 is chosen so small that first $|f - p_\alpha| = |g - \alpha[p_0 + \kappa_1 h''(\varphi^{-1}(\alpha)) x]|$ decreases as x moves away from r_i in a neighborhood of $S = \{r_1, r_2, r_3\}$ for all α $(0 < \alpha \leqslant \alpha_0)$. This can be done since $|g|$ strictly decreases linearly as x moves away from each r_i. Hence α_0 can be chosen so small that $\|f - p_\alpha\| = \max_{x \in S} |f - p_\alpha|$, $0 < \alpha \leqslant \alpha_0$. Thus $\|f - p_\alpha\| = 1 + |\kappa_1 r_*|\ \alpha h''(\varphi^{-1}(\alpha))$ for some $r_* \in \{r_1, r_2, r_3\}$. Also, note that $\|f - p_f\| = \|g\| = 1$ and $\|p_f - p_\alpha\| \geqslant |p_f(0) - p_\alpha(0)| = |c_0|\, \alpha$. Furthermore, $p_\alpha'(x) = \kappa h'(x) + \alpha[h'(x) + \kappa_1 h''(x)] + \kappa_1 \alpha h''(\varphi^{-1}(\alpha))$. By replacing p_0 by $-p_0$ if necessary, we may assume κ_1 is positive. Then for $x > 0, p_\alpha'(x) > 0$; for $x \in [r_1, -\varphi^{-1}(\alpha)]$ and κ chosen sufficiently large (initially), since $h'(x) = \lambda(x) \varphi(x) h''(x)$, where $\lambda(x) \to \lambda > 0$, $x \to 0^+$, $\kappa h'(x)$ dominates $\kappa_1 \alpha h''(x)$ showing that $p_\alpha'(x) > 0$ here; for $x \in [-\varphi^{-1}(\alpha), 0]$, $h''(\varphi^{-1}(\alpha)) \geqslant |h''(x)|$, again implying that $p_\alpha'(x) \geqslant 0$. Thus $p_\alpha \in M_4$ and $(\|f - p_\alpha\| - \|f - p_f\|)/u(\|p_\alpha - p_f\|) \leqslant |\kappa_1 r_*|\, \alpha h''(\varphi^{-1}(\alpha))/u(|c_0|\, \alpha)$. Thus $u(x) = x h''(\varphi^{-1}(|c_0|^{-1} x))$ is the best rate function that could hold in (2), and the proof of Theorem 1 is complete as soon as we indicate how f can be chosen monotone. Note, however, that as long as κ is large enough $f = g + \kappa h$ is admissible. Also, for κ large enough, since h is odd and monotone with $h'(x) > 0$ except at $x = 0$, $\kappa h'$ will dominate g' outside the neighborhood $(-\varepsilon, \varepsilon)$ of $x = 0$, prescribed at the beginning of the proof, and thus $g + \kappa h$ will be monotone there. On the other hand, in $(-\varepsilon, \varepsilon)$ $f' = \kappa h' \geqslant 0$. Thus for κ large enough f satisfies the restraints (i.e., f is monotone).

Next we show that, under the additional hypotheses of Theorem 2, for the above f and p_f, (2) does in fact hold with $u(x) = x\psi(y)\, h''(y)$, $y = \varphi^{-1}(cx)$

for some constant $c > 0$. Let $E = E^0 \cup E^1$, where $E^0 = \{e_{r_i}\}_{i=1}^3$ and $E^1 = \{e_0'\}$. Define the semi-norm $\|\cdot\|'$ on V_4 by $\|q\|' = \max\{|e(q)| : e \in E\}$. Set $Q = \{q = (p_f - p)/\|p_f - p\|' : \|p_f - p\|' \neq 0$ and $p \in M_4\}$. We claim that $\inf_{q \in Q} \max_{e \in E^0} \sigma(e)\, e(q) = \tau > 0$, where $\sigma(e_{r_i}) = \operatorname{sgn}(f(r_i) - p_f(r_i)) = (-1)^i$, $i = 1, 2, 3$, and $\sigma(e_0') = 1$. Indeed, suppose there exists $q_m \in Q$ for which $\lim_{m \to \infty} \max_{e \in E^0} \sigma(e)\, e(q_m) \leqslant 0$. Also, from $q_m = (p_f - p_m)/\|p_f - p\|'$ with $\|p_f - p\|' \neq 0$ and $p \in M_4$, we see that $\sigma(e)\, e(q_m) \leqslant 0$ for $e \in E^1$. Thus $\lim_{m \to \infty} \sigma(e)\, e(q_m) \leqslant 0$ for all $e \in E$ and hence, since 0 belongs to the convex hull of $\{\sigma(e)\, e : e \in E\}$, we conclude that $\lim_{m \to \infty} e(q_m) = 0$ $\forall e \in E$. Hence $\lim_{m \to \infty} \|q_m\|' = 0$ while $\|q_m\|' = 1$, a contradiction. Hence there exists $e \in E^0$ for which $\sigma(e)\, e(p_f - p) \geqslant \tau \|p_f - p\|'$. Now observe that $\|f - p\| \geqslant \sigma(e)(e(f) - e(p)) = \sigma(e)(e(f) - e(p_f)) + \sigma(e)(e(p_f) - e(p)) = \|f - p_f\| + \sigma(e)(e(p_f) - e(p)) \geqslant \|f - p_f\| + \tau \|p_f - p\|'$. Observing that this inequality is also true if $\|p_f - p\|' = 0$, we have established a strong uniqueness-type result with the seminorm $\|\cdot\|'$. Next, a second norm is introduced; namely, $\|p\|^* = \max\{|e(p)| : e \in E^{aug}\}$, where $E^{aug} = E \cup \{l_3\}$, where l_3 is the augmented extremal discussed above. That $\|\ \|^*$ is in fact a norm on V_4 is immediate from the lemma. Thus, there exists a constant $\gamma' > 0$ such that $\|p\|^* \geqslant \gamma' \|p\|$ for all $p \in V_4$. Finally, we wish to establish that there exist $A > 0$ and $\kappa > 0$ for which $\|p_f - p\|' \geqslant A u(\kappa \|p_f - p\|^*)$ for all $p \in M_4$ satisfying $\|p\| \leqslant N$. First observe that if $\|p_f - p\|' = 0$, then since $p \in M_4$ we have that $e(p_f - p) = 0$ for all $e \in E^{aug}$, implying that $\|p_f - p\|^* = 0$ or $p_f = p$. Now, for $e \in E$, we clearly have that for any $\kappa > 0$ there exists a constant K_1 for which $|e(p_f - p)| \geqslant K_1 u(\kappa |e(p_f - p)|)$ since $\|p\| \leqslant N$, where $u(x) = x \psi(y)\, h''(y)$ with $y = \varphi^{-1}(x/|c_0|)$, as defined above. Let $e = l_3$. We claim that there exist $K_2 > 0$ and $\kappa > 0$ for which $|e_0'(p_f - p)| \geqslant K_2 u(\kappa |l_3(p_f - p)|)$ for all $p \in M_4$ satisfying $\|p\| \leqslant N$. Suppose that this is not the case. Then, for any fixed $\kappa > 0$, corresponding to each integer $\nu > 0$ there exists $q_\nu \in M_4$ with $\|q_\nu\| \leqslant N$ for which $|q_\nu'(0)| < (1/\nu)\, u(\kappa |l_3(q_\nu)|)$. By passing to subsequences if necessary we may assume that q_ν converges uniformly to $q \in M_4$. Clearly, we must have $q'(0) = 0$. We can write $q_\nu'(x) = q_\nu'(0) + l_3(q_\nu)\, h'' + c_\nu h' = \beta_\nu + \alpha_\nu h'' + c_\nu h'$, where $\beta_\nu \geqslant 0$, $\beta_\nu \to 0$ (since $q'(0) = 0$), $\alpha_\nu \neq 0$, $\alpha_\nu \to 0$ (since $l_3(q) = 0$ because $q \in M_4$ and $q'(0) = 0$), $c_\nu \to c$, and $q_\nu'(x) \geqslant 0$, $\forall x \in [a, b]$; note $q = q(0) + ch$. Note also that since $(1, x, h', h)$ is a basis for V_4, if $p \in V_4$ and $\|p\| \leqslant N$, then the coefficient of h in the expansion for p must be bounded above by some constant c_* depending only on N. Thus $\forall x \in [a, b]$, $q_\nu^*(x) = \beta_\nu + \alpha_\nu h'' + c_* h' > 0$, where $|c_\nu| < c_*$, $\forall \nu$. Now q_ν^* has a critical point in $[a, b]$ for ν sufficiently large as follows: $q_\nu^{*\prime}(x) = \alpha_\nu h'''(x) + c_* h''(x) = 0$ has a solution $x_\nu = x_\nu(\alpha_\nu)$ for α_ν sufficiently small since $h''(x)/h'''(x) = (\operatorname{sgn} x)(\mu + \delta_1(|x|))\, \varphi(|x|)$, $\delta_1(x) = o(1)$; here $\lambda \leqslant \mu < \infty$ since, by L'Hospital's rule $h''\varphi/h' = h'''\varphi/h'' + \varphi' + o(1)$, and $h'(x)/h''(x) = (\operatorname{sgn} x)(\lambda + \delta_2(|x|))\, \varphi(|x|)$, where $\delta_2(x) = o(1)$, and $\varphi'(0) \geqslant 0$. In fact then $|x_\nu| = \varphi^{-1}(-(\operatorname{sgn} x_\nu)\, \alpha_\nu /$

$(\mu+\delta_1(|x_\nu|))\,c_*)=\varphi^{-1}(|\alpha_\nu|/(\mu+\delta_1(|x_\nu|))\,c_*)$. Now choose $0<\kappa<(\mu c_*)^{-1}$. Thus, for ν sufficiently large, $|x_\nu|>\varphi^{-1}(\kappa|\alpha_\nu|)$ and

$$\begin{aligned}\beta_\nu &> -\alpha_\nu h''(x_\nu)-c_* h'(x_\nu)\\ &= [-\alpha_\nu - c_*(\operatorname{sgn} x_\nu)(\lambda+\delta_2(|x_\nu|))\,\varphi(|x_\nu|)]\,h''(x_\nu)\\ &= c_*(\operatorname{sgn} x_\nu)[\mu+\delta_1(|x_\nu|)-\lambda-\delta_2(|x_\nu|)]\,\varphi(|x_\nu|)\,h''(x_\nu)\\ &= c_*(\tau+o(x_\nu))\,\psi(|x_\nu|)\,\varphi(|x_\nu|)\,h''(|x_\nu|),\end{aligned}$$

for some positive constant τ, from the definition of ψ. Then since $\varphi(|x_\nu|)\geqslant\kappa|\alpha_\nu|$ and $|x_\nu|\geqslant y_\nu=\varphi^{-1}(\kappa|\alpha_\nu|)$, and since ψ,φ, and h'' are nondecreasing, the preceding inequality leads to

$$\beta_\nu > c_*\tau'\kappa|\alpha_\nu|\,\psi(y_\nu)\,h''(y_\nu),$$

where $0<\tau'<\tau$ and ν is sufficiently large, which is our desired contradiction. We conclude by setting $c=\kappa\gamma'$ (in the statement of Theorem 2). ▮

Applications

EXAMPLE 2 ($h=xe^{-x^{-2}}$). We show that the hypotheses of Theorem 1 hold. First V_4 is Haar in some neighborhood of the origin. To see this, note that $(h',h'',h''')=((x^2+2)/x^2,\ 2(2-x^2)/x^5,\ 2(3x^4-12x^2+4)/x^8)\ e^{-x^{-2}}$ and apply part (ii) of the lemma below. The remaining hypotheses of Theorem 1 are easily checked and we can take $\varphi(x)=x^3$. We conclude that the rate of strong unicity is at best $u(x)=x^{-2/3}e^{-c_1x^{-2/3}}$ for some constant $c_1>0$. In particular, we have an example where the best approximation is unique but the "order" $\alpha=0$; in fact then any rate function decays at best exponentially.

Further, however, the additional hypotheses of Theorem 2 are seen to hold where $\varphi'(0)=0$ and $\psi(y)=\frac{3}{4}y^2$ is asymptotic to $((h''/h''')-(h'/h''))/\varphi$, as is easily checked. Hence (2) holds with $u(x)=e^{-c_2x^{-2/3}}$ for some constant $c_2>0$. We conclude that the best possible rate function u satisfies $e^{-c_2x^{-2/3}}\leqslant u(x)\leqslant x^{-2/3}e^{-c_1x^{-2/3}}$ for constants $0<c_1<c_2$ and thus decays exponentially.

EXAMPLE 3 ($h=(\operatorname{sgn} x)|x|^{2+r}, r>0$). Note that if r is an odd integer, then $h=x^{2+r}$ and we are in the case of Example 1. One can check immediately that all the hypotheses of Theorems 1 and 2 hold except for the Haar hypothesis on V_4. But to see that V_4 is Haar on $(-\infty,\infty)$, apply part (i) of the lemma below (if $r\geqslant1$ also, (ii) applies). As in Example 1, $\varphi(x)=x$ and we conclude that (2) holds with $u(x)=[1/(2+r)(1+r)]$

$xh''(x) = x^{r+1}$ and u is best possible. In other words this example provides strong uniqueness of arbitrary "order" $\alpha = [1/(1+r)] \in (0, 1)$.

LEMMA. *Let* $V_4 = [1, x, h'(x), h(x)]$, *where* h *is odd in* $(-a, a)$, $h \in C^2(-a, a)$, $h'(0) = 0$, *and* h'' *is strictly increasing. Then* V_4 *is Haar on* $(-a, a)$ *if on* $(-a, a)$

(i) $h' = \kappa |h''|^\rho$ *for some* $\rho > 1$ *and* $\kappa > 0$, *or*

(ii) $h \in C^3(-a, a)$, h''/h''' *is strictly monotonic, and* $\lim_{x\to 0} (h''(x)/h'''(x)) = 0$.

Proof. Show $(V_4)' = [1, h'', h']$ is Haar in both cases by considering the Vandermonde determinant

$$D = \begin{vmatrix} 1 & h''(x_1) & h'(x_1) \\ 1 & h''(x_2) & h'(x_2) \\ 1 & h''(x_3) & h'(x_3) \end{vmatrix}.$$

In case (i) let $y = h''(x)$; then

$$D = \kappa \begin{vmatrix} 1 & y_1 & |y_1|^\rho \\ 1 & y_2 & |y_2|^\rho \\ 1 & y_3 & |y_3|^\rho \end{vmatrix}$$

$$= \kappa (y_2 - y_1)(y_3 - y_2) \left[\left(\frac{|y_3|^\rho - |y_2|^\rho}{y_3 - y_2} \right) - \left(\frac{|y_2|^\rho - |y_1|^\rho}{y_2 - y_1} \right) \right].$$

Hence $(|y_{i+1}|^\rho - |y_i|^\rho)/(y_{i+1} - y_i) = \rho (\operatorname{sgn} \eta_i) |\eta_i|^{\rho - 1}$, $i = 1, 2$, where $y_1 < \eta_1 < y_2 < \eta_2 < y_3$; and so $D \neq 0$ since $f(\eta) = (\operatorname{sgn} \eta) |\eta|^{\rho-1}$ is an increasing function.

In case (ii),

$$D = \kappa(x_1, x_2, x_3) \left(\frac{h'(x_3) - h'(x_2)}{h''(x_3) - h''(x_2)} - \frac{h'(x_2) - h'(x_1)}{h''(x_2) - h''(x_1)} \right),$$

where $\kappa(x_1, x_2, x_3) = (h''(x_2) - h''(x_1))(h''(x_3) - h''(x_2))$. Hence $D = \kappa(x_1, x_2, x_3)(h''(\eta_2)/h'''(\eta_2) - h''(\eta_1)/h'''(\eta_1))$, where $x_1 < \eta_1 < x_2 < \eta_2 < x_3$; so $D \neq 0$ by hypothesis. (Note that $h''' > 0$ in $(-a, a)$ except possibly at $x = 0$. If $h'''(0) = 0$ and $0 \in (x_i, x_{i+1})$, then the mean value theorem holds for $(h'(x_{i+1}) - h'(x_i))/(h''(x_{i+1}) - h''(x_i))$ as follows: First, if $h'(x_{i+1}) - h'(x_i) \neq 0$, let $h_\varepsilon'''(x) = h'''(x) + \varepsilon$, $\varepsilon > 0$, and $h_\varepsilon''(x) = h''(x) + \varepsilon x$, $h_\varepsilon'(x) = h'(x) + \varepsilon x^2/2$. Then $(h_\varepsilon'(x_{i+1}) - h_\varepsilon'(x_i))/(h_\varepsilon''(x_{i+1}) - h_\varepsilon''(x_i)) = h_\varepsilon''(\xi_\varepsilon)/h_\varepsilon'''(\xi_\varepsilon)$. Then let $\varepsilon \to 0$ and let ξ be a subsequential limit point of ξ_ε (note that $\xi \neq 0$);

thus $(h'(x_{i+1})-h'(x_i))/(h''(x_{i+1})-h''(x_i))=h''(\xi)/h'''(\xi)$. Secondly, if $h'(x_{i+1})-h'(x_i)=0$, then $(h'(x_{i+1})-h'(x_i))/(h''(x_{i+1})-h''(x_i))=0=\lim_{x\to 0}(h''(x)/h'''(x))=h''(0)/h'''(0)$ (by implicit definition).) ∎

COROLLARY. *Given* $u\in C^2[0,\alpha)$, $u(0)=u'(0)=0$, $u(x)/x$ *increasing*, $\lim_{x\to 0}(xu'/u)>1$, $\int_0^\alpha (xu'/u)<\infty$, *and* $u'/u\geqslant(u''/u')+(1/x)$, *then there exists a problem of best monotone approximation from a Haar space with rate of strong uniqueness at best* $u(c_1x)$ *and at least* $\psi(\varphi^{-1}(x))\,u(c_2x)$, *where* $\varphi^{-1}(x)=\int_0^x (tu'(t)/u(t))\,dt$, $\psi=\varphi'/(1-\varphi')$, *and* c_1, c_2 *are positive constants.*

Proof. Let $h(x)=\int_0^x (u[\varphi(t)]/u[\varphi(\alpha)])\,dt$, $0\leqslant x<\alpha$, and extend h oddly to $-\alpha<x<0$. Then check that all the hypotheses of Theorems 1 and 2 (including part (ii) of the lemma above) are satisfied. Next apply the conclusions of Theorems 1 and 2 to obtain the desired conclusion. ∎

EXAMPLE 4. $(u(x)=e^{-x^{-s}},\ 1>s>0)$.

EXAMPLE 5. $(u(x)=x^{1+s},\ s\geqslant 1)$.

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